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INVESTIGATION OF GASDYNAMIC PROCESSES BY THE METHOD OF TRAVELING WAVES WITH ALLOWANCE FOR VOLUMETRIC SOURCES AND SINKS

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Using the method of traveling waves, we investigate the effect of volumetric sources and sinks of mass and energy on the motion of gas. We assume that the strengths of the sources are power functions of the temperature and density. The solution of the corresponding system of ordinary differential equations is constructed analytically: the sought functions are represented in explicit or implicit form. It is shown that the presence of sources or sinks leads to substantially new properties of the solutions of the equations of gas dynamics. Depending on the initial parameters of the problem, the solution may exist both at any instant of time $t > 0$ and only in a finite interval of t.

Introduction. In the study of motion and heat transfer in a continuous medium problems of gas dynamics with allowance for different kinds of nonlinear volumetric sources and sinks are of great interest. We know, for example, what role in the heating and compression of a high-temperature plasma is played by energy release due to laser-radiation absorption, volumetric losses of energy in intrinsic thermal radiation, and energy release from thermonuclear reactions. The heating and compression of plasma by an axial magnetic field (theta-pinch) are influenced greatly by mass losses through the end faces of the plasma pinch and end-face energy losses due to longitudinal electronic heat conduction [1-6].

The present work is devoted to mathematical simulation of the motion of a gas with allowance for volumetric mass and energy sources and sinks. This simulation includes not only development of corresponding numerical methods and setting up of programs and numerical calculations, but also a preliminary qualitative analysis that makes it possible to elucidate the characteristic features of the process investigated. Different analytical methods are used for this purpose, including techniques of self-similar and other types of invariant-group solutions (of the traveling-wave type, exponential type, and others). Results of investigations of self-similar motion of gas in the presence of nonlinear mass and energy sources and sinks and volumetric forces in the medium are presented in monograph [7] (see also the bibliography to that work). In [3, 6, 8] different techniques (different models) of allowance for mass and energy losses in a moving medium are considered. It is assumed in one of the models that particles that leave an element of the flow carry away energy. This assumption is used, for example, in the case where a change in mass in an element of flow is associated with its transfer by products of thermonuclear reactions [9, 10]. Another model takes into consideration that escaping particles have a corresponding energy and, moreover, do the work against pressure forces. Therefore, the particles carry away enthalpy. This model was used to describe mass losses from the end faces of a plasma pinch in compression and heating of plasma in theta-pinches [1-6].

We investigate the influence of volumetric sources or sinks on the motion of gas by the method of traveling waves. Just as in [11-14], we assume that traveling waves propagate in the medium due to the effect of a piston. We carry out investigation assuming the validity of the equations of state of an ideal gas and a power dependence of the strength of sources and sinks on thermodynamic quantities. We show that the presence of sources and sinks leads to substantially new properties of the solutions of the equations of gas dynamics. At certain values of the

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problem parameters a solution of the traveling-wave type exists only for a finite time, as is the case in a heat-conducting medium without regard for sources and sinks [11-15]. At other values of the parameters, similarly to the case where only energy sources and sinks are taken into account in a nonconducting medium [7], the solution may exist at any instant of time $t > 0$.

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1. Statement of the Problem. A system of equations of gas dynamics with allowance for volumetric mass and energy sources or sinks under the assumption that the velocity of the particles arriving at a flow element or escaping from it coincides with the velocity of the gas can be written for the case of plane symmetry in the following form [4, 6]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho v) = -\rho\chi, \tag{1.1}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial r}(\rho v^2) + \frac{\partial P}{\partial r} = -\rho\chi v, \tag{1.2}$$

$$\frac{\partial}{\partial t}\left[\rho\left(\varepsilon + \frac{v^2}{2}\right)\right] + \frac{\partial}{\partial r}\left[\rho v\left(\varepsilon + \frac{P}{\rho} + \frac{v^2}{2}\right)\right] = G - \rho\chi\left(\varepsilon + \frac{v^2}{2} + \nu_0\frac{P}{\rho}\right). \tag{1.3}$$

Here, $\nu_0 = 1$ if the particles carry away or bring enthalpy, and $\nu_0 = 0$ when this is energy. The thermal conductivity of the medium is not taken into account.

Similarly to [4, 7, 16, 17], it is convenient to select, as the space coordinate, the parameter q, which is defined by the initial distribution of mass:

$$q = m(0) = \int_{r_0(0)}^{r(0)} \rho(y, 0)\, dy. \tag{1.4}$$

We pass from the Euler variables r, t to the quasi-Lagrangian coordinates q and t_L by the formulas

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_L} - \frac{\rho v}{\psi}\frac{\partial}{\partial q}, \quad \frac{\partial}{\partial r} = \frac{\rho}{\psi}\frac{\partial}{\partial q}, \tag{1.5}$$

where the function $\psi = \psi(q, t) = \partial m/\partial q$ determines the fraction of the mass left in, or acquired by, the given element of the flow.

Omitting the subscript L at t, we write the system of equations (1.1)-(1.3) in the following form:

$$\frac{\partial \psi}{\partial t} = -\chi\psi. \tag{1.6}$$

$$\frac{\partial}{\partial t}\left(\frac{\psi}{\rho}\right) = \frac{\partial v}{\partial q}, \tag{1.7}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\psi}\frac{\partial P}{\partial q}, \tag{1.8}$$

$$\frac{\partial \varepsilon}{\partial t} = -P\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) + \frac{G}{\rho} + \chi(1 - \nu_0)\frac{P}{\rho}, \tag{1.9}$$

We will assume the equations of state of an ideal gas to be valid:

$$P = R_0 \rho T\,, \quad \varepsilon = \frac{R_0}{\gamma - 1} T = \frac{1}{\gamma - 1} \frac{P}{\rho}\,, \tag{1.10}$$

where R_0 is the gas constant; $\gamma > 1$ is the constant ratio of the specific heat capacities.

The strengths of the sources or sinks χ and G will be assumed to be power functions of thermodynamic quantities:

$$\chi = \chi_0 T^{a_0} \rho^{b_0}\,, \quad G = G_0 T^{a_1} \rho^{b_1}\,, \tag{1.11}$$

where $\chi_0 < 0$, $G_0 > 0$ for sources of mass and energy, and $\chi_0 > 0$, $G_0 < 0$ for their sinks.

Sometimes it is convenient to consider the energy equation (1.9) in an entropy form, introducing the so-called "entropy" function

$$\Sigma = \varepsilon \rho^{-(\gamma-1)} = \frac{R_0}{\gamma - 1} T \rho^{-(\gamma-1)}\,. \tag{1.12}$$

We represent Eq. (1.9) as

$$\frac{\partial \Sigma}{\partial t} = \rho^{-\gamma} G + (\gamma - 1)(1 - \nu_0) \chi \Sigma\,. \tag{1.13}$$

Using Eq. (1.12), we can represent formulas (1.11) in the form

$$\begin{aligned} \chi &= \chi_0 (\gamma - 1)^{a_0} R_0^{-a_0} \Sigma^{a_0} \rho^{a_0(\gamma-1)+b_0}\,, \\ G &= G_0 (\gamma - 1)^{a_1} R_0^{-a_1} \Sigma^{a_1} \rho^{a_1(\gamma-1)+b_1}\,. \end{aligned} \tag{1.14}$$

The dimensions of the quantities G, χ, and P are interrelated as $[G] = [\chi P]$. We will assume below that

$$G = \lambda_0 \chi P\,, \tag{1.15}$$

where λ_0 is a dimensionless constant, with $\lambda_0 = G_0/(\chi_0 R_0)$ at $\chi_0 \neq 0$. It follows from (1.11) that in the case of (1.15) the following conditions are satisfied:

$$a_1 = a_0 + 1\,, \quad b_1 = b_0 + 1\,. \tag{1.16}$$

In the case of (1.15), using (1.6) and (1.11), Eq. (1.13) can be written as

$$\frac{\partial}{\partial t} \left(\Sigma \psi^{(\gamma-1)(1-\nu_0+\lambda_0)}\right) = 0\,. \tag{1.17}$$

The gasdynamic and thermal quantities that describe traveling waves depend on the coordinates q and t not separately, but in the combination $q - Dt$, where D is a constant: any sought function can be represented in the form $F(q, t) = F(q - Dt)$. We formulate the following problem. Let a plane piston move into a nonconducting gas with volumetric mass and energy sources or sinks. In this case the boundary condition on the piston is specified in such a way that a traveling wave moves in front of the piston. We will determine below how to specify such a regime on the piston. We will characterize the interface between the gas and the piston by the coordinate $q = 0$. The gas is located in the region $q > 0$.

Let the following condition be assigned at $q = 0$:

$$v(0, t) = v_*(t) > 0\,. \tag{1.18}$$

Next, when $t = 0$ and $t > 0$ let there be no sources or sinks ahead of the traveling-wave front and the gas be cool and stagnant and have the constant initial density $\rho = \rho_0 = \text{const}$, i.e., let the following conditions be satisfied:

$$\psi \equiv 1, \quad T = 0, \quad v = 0, \quad \rho = \rho_0. \tag{1.19}$$

The independent variables and functions that satisfy system of equations (1.6)-(1.12), (1.14) can be represented in the following dimensionless form:

$$s = \frac{Dt - q}{M_0}, \quad \eta = \eta(s) = \frac{\rho_0}{\rho(q, t)}, \quad \alpha = \alpha(s) = \frac{v(q, t)}{D\rho_0^{-1}},$$

$$\beta = \beta(s) = \frac{P(q, t)}{D^2\rho_0^{-1}}, \quad f = f(s) = \frac{R_0 T(q, t)}{D^2\rho_0^{-2}}, \quad \hat{\chi} = \hat{\chi}(s) = \frac{\chi(q, t)}{DM_0^{-1}}, \tag{1.20}$$

$$\hat{G} = \hat{G}(s) = \frac{G(q, t)}{D^3 M_0^{-1}\rho_0^{-1}}, \quad \xi = \xi(s) = \frac{\Sigma(q, t)}{D^2\rho_0^{-(\gamma+1)}}, \quad \varphi = \varphi(s) = \psi(q, t).$$

Here η is the reciprocal of the density; it is the specific volume written in dimensionless form; $M_0 = |\chi_0^{-1}| R_0^{a_0} D^{-2a_0+1} \rho_0^{2a_0-b_0}$ is a dimensional constant.

Let us go over to variables (1.20) using the following differential relations:

$$\frac{\partial}{\partial t} = \frac{D}{M_0}\frac{d}{ds}, \quad \frac{\partial}{\partial q} = -\frac{1}{M_0}\frac{d}{ds}.$$

Denoting d/ds by a prime, we obtain from the initial system of partial differential equations (1.6)-(1.9), with allowance for (1.10), the following system of ordinary differential equations in one independent variable s:

$$\varphi' = -\hat{\chi}\varphi, \tag{1.21}$$

$$(\varphi\eta)' = -\alpha', \tag{1.22}$$

$$\alpha' = \frac{1}{\varphi}\beta', \quad \beta = f\eta^{-1}, \tag{1.23}$$

$$\frac{1}{\gamma - 1}f' = -\beta\eta' + \hat{G}\eta + (1 - \nu_0)\hat{\chi}\beta\eta, \tag{1.24}$$

where

$$\hat{\chi} = \hat{\chi}_0 f^{a_0}\eta^{-b_0} = \hat{\chi}_0(\gamma - 1)^{a_0}\xi^{a_0}\eta^{-a_0(\gamma-1)-b_0},$$
$$\hat{G} = \hat{G}_0 f^{a_0+1}\eta^{-(b_0+1)} = \hat{G}_0(\gamma - 1)^{a_0+1}\xi^{a_0+1}\eta^{-a_0(\gamma-1)-\gamma-b_0}. \tag{1.25}$$

We will assume that the condition $q = Dt$ corresponds to the traveling-wave front and, consequently, $s = 0$. The perturbed region is located in the interval $s \geq 0$. Conditions (1.19), which must be satisfied when $s \leq 0$, take the following form in variables (1.20):

$$\varphi = 1, \quad f = 0, \quad \alpha = 0, \quad \eta = 1. \tag{1.26}$$

In the case where the piston moves into the gas, a compression wave propagates ahead of it, which can have a strong discontinuity of shock-wave type moving over the background (1.19) (see [15, 18, 19]).

We denote values of the sought functions behind the shock-wave front by the subscript 1:

$$f = f_1\,, \quad \alpha = \alpha_1\,, \quad \eta = \eta_1\,. \tag{1.27}$$

The function φ is continuous:

$$\varphi = \varphi_1 = 1\,. \tag{1.28}$$

In the presence of a discontinuity, system of equations (1.21)-(1.25) must satisfy conditions (1.27) and (1.28) at $s = 0$. The values (1.27) are connected with (1.26) by relations that follow from conservation laws at the discontinuity front (by Hugoniot conditions [15, 19]). With allowance for (1.19), these conditions are represented as follows in initial (dimensional) variables:

$$\rho = \rho_1 = \frac{\gamma + 1}{\gamma - 1}\rho_0\,, \quad v = v_1 = \frac{2}{\gamma + 1} D\rho_0^{-1}\,,$$

$$P = P_1 = \frac{2}{\gamma + 1} D^2\rho_0^{-1}\,, \quad T = T_1 = \frac{1}{R_0}\frac{2(\gamma - 1)}{(\gamma + 1)^2} D^2\rho_0^{-2}\,, \tag{1.29}$$

$$\Sigma = \Sigma_1 = \frac{2}{(\gamma + 1)^2}\left(\frac{\gamma - 1}{\gamma + 1}\right)^{\gamma - 1} D^2\rho_0^{-(\gamma+1)}\,, \quad \psi = \psi_1 = 1\,.$$

In variables (1.20) conditions (1.29) take the form

$$\eta_1 = \frac{\gamma - 1}{\gamma + 1}\,, \quad \alpha_1 = \frac{2}{\gamma + 1}\,, \quad \beta_1 = \frac{2}{\gamma + 1}\,,$$

$$f_1 = \frac{2(\gamma - 1)}{(\gamma + 1)^2}\,, \quad \xi_1 = \frac{2}{\gamma^2 - 1}\left(\frac{\gamma - 1}{\gamma + 1}\right)^{\gamma}\,, \quad \varphi_1 = 1\,. \tag{1.30}$$

Equation (1.22) can be integrated. Taking into account conditions (1.27), (1.28), and (1.30), we obtain

$$\alpha = 1 - \varphi\eta\,. \tag{1.31}$$

Now, we determine the possible form of function (1.18) specified on the piston. By assumption, the interface between the gas and the piston is the coordinate $q = 0$. From (1.20) it follows that the dimensionless independent variable s is determined at $q = 0$ by the formula $s = Dt/M_0$. Therefore, the function $\alpha = \alpha(s)$ at $s = Dt/M_0$ will describe in dimensionless form the gasdynamic regime specified on the piston. Solving system of equations (1.21)-(1.25) with boundary conditions of the form (1.27), (1.28) at the point $s = 0$ to determine the function $\alpha = \alpha(s) = \alpha(Dt/M_0)$, we find the corresponding expression for the function $v_*(t)$ in formula (1.18):

$$v_*(t) = \alpha\left(\frac{D}{M_0}t\right) D\rho_0^{-1} = \left[1 - \varphi\left(\frac{D}{M_0}t\right)\eta\left(\frac{D}{M_0}t\right)\right] D\rho_0^{-1}\,.$$

It is evident that the solutions considered have physical meaning only in the case where the variable s increases starting from the value $s = 0$, because the time $t > 0$ should increase.

Equation (1.17) takes the following form in variables (1.20):

$$\left[\xi\varphi^{(\gamma-1)(1-\nu_0+\lambda_0)}\right]' = 0\,. \tag{1.32}$$

Integrating (1.32), we obtain

$$\xi\varphi^{(\gamma-1)(1-\nu_0+\lambda_0)} = C_0\,. \tag{1.33}$$

When $s = 0$, the sought functions must satisfy conditions (1.27) and (1.28), where the parameters with subscript 1 are defined by formulas (1.30). From the indicated conditions we have

$$C_0 = \xi_1 = \frac{2}{\gamma^2 - 1}\left(\frac{\gamma - 1}{\gamma + 1}\right)^{\gamma}.$$

We write formula (1.33) as

$$\xi = \xi_1 \varphi^{-(\gamma-1)(1-\nu_0+\lambda_0)}. \tag{1.34}$$

Formula (1.34) is valid in the region $s \geq 0$. When $s < 0$, we must have $\xi \equiv 0$. The dimensionless pressure β and temperature f functions can be expressed in terms of the specific volume η and the entropy function ξ:

$$\beta = (\gamma - 1)\,\xi\eta^{-\gamma}, \quad f = (\gamma - 1)\,\xi\eta^{-\gamma+1}. \tag{1.35}$$

Using (1.34), (1.35), and expression (1.25) for $\hat{\chi}$, we reduce the problem considered to the solution of two ordinary differential equations of the form

$$\varphi' = -\hat{\chi}_0(\gamma - 1)^{a_0}\xi_1^{a_0}\varphi^{-a_0(\gamma-1)(1-\nu_0+\lambda_0)+1}\eta^{-a_0(\gamma-1)-b_0}, \tag{1.36}$$

$$\varphi\left[\eta^{\gamma+1} - (\gamma - 1)\,\xi_1\gamma\varphi^{-(\gamma-1)(1-\nu_0+\lambda_0)-2}\right]\eta' =$$
$$= -\eta\left[\eta^{\gamma+1} - (\gamma - 1)^2(1 - \nu_0 + \lambda_0)\,\xi_1\varphi^{-(\gamma-1)(1-\nu_0+\lambda_0)-2}\right]\varphi'. \tag{1.37}$$

The solution of Eq. (1.36) and (1.37) must satisfy the conditions

$$\varphi(0) = 1, \quad \eta(0) = \eta_1 = \frac{\gamma - 1}{\gamma + 1}, \tag{1.38}$$

The solution is sought in the region $s \geq 0$. When $s < 0$, we have

$$\varphi \equiv 1, \quad \eta \equiv 1. \tag{1.39}$$

There is a strong discontinuity at the point $s = 0$.

2. Influence of Sources and Sinks on the Distribution of the Entropy Function $\xi = \xi(s)$ and the Function $\varphi = \varphi(s)$. Below we assume

$$a_0(\gamma - 1) + b_0 = 0. \tag{2.1}$$

Under condition (2.1) Eq. (1.36) can be integrated. Let us consider various cases.

a) $a_0(1 - \nu_0 + \lambda_0) = 0$. The solution of Eq. (1.36) under condition (2.1) with allowance for the condition $\varphi(0) = 1$ has the form

$$\varphi = \exp(-\hat{\chi}_0(\gamma - 1)^{a_0}\xi_1^{a_0}s). \tag{2.2}$$

b) $a_0(1 - \nu_0 + \lambda_0) \neq 0$. The solution of Eq. (1.36) is determined by the formula

$$\varphi = \left[1 - \hat{\chi}_0(\gamma - 1)^{a_0+1}\xi_1^{a_0}(1 - \nu_0 + \lambda_0)\,s\right]^{1/[a_0(\gamma-1)(1-\nu_0+\lambda_0)]}. \tag{2.3}$$

The qualitative character of the distribution of functions (2.2) and (2.3) is depicted in Fig. 1.

In the case of volumetric losses of mass ($\hat{\chi}_0 = 1$, $\lambda_0 = \hat{G}_0$), we obtain that, when $\hat{G}_0 < \nu_0 - 1$, the function $\varphi = \varphi(s)$ decreases with increase in s and attains the value $\varphi(\infty) = 0$ at $s = \infty$ (see the solid line in Fig. 1b). When $\hat{G}_0 > \nu_0 - 1$, with increase in s from $s = 0$ the function $\varphi = \varphi(s)$ vanishes at a certain finite value

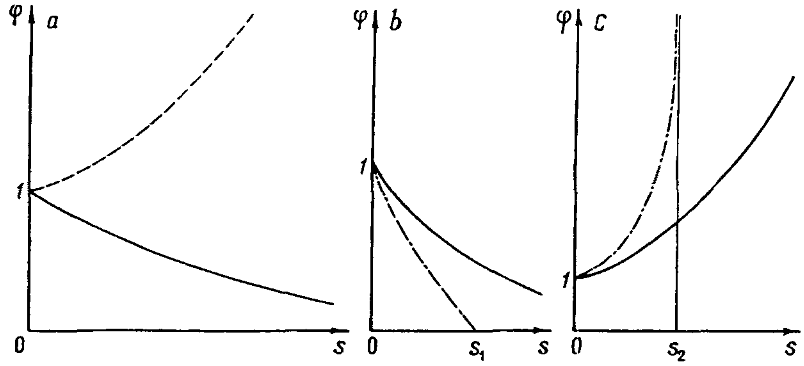


Fig. 1. Distribution of the function $\varphi = \varphi(s)$ at $\hat{\chi}_0 = 1$ (mass sink) and $\hat{\chi}_0 = -1$ (mass inflow): a) $a_0(1 - \nu_0 + \lambda_0) = 0$ at $\hat{\chi}_0 = 1$ (solid line) and $\hat{\chi}_0 = -1$ (dashed line); b) and c) $\hat{G}_0 < \nu_0 - 1$ at $\hat{\chi}_0 = 1$ and $\hat{G}_0 < 1 - \nu_0$ at $\hat{\chi} = -1$ (solid line), $\hat{G}_0 > \nu_0 - 1$, $\hat{\chi}_0 = 1$ and $\hat{G}_0 > 1 - \nu_0$, $\hat{\chi}_0 = -1$ (dash-dot line).

$$s = s_1 = 1/[(\gamma - 1)^{a_0+1} \xi_1^{a_0} (1 - \nu_0 + \hat{G}_0)] . \tag{2.4}$$

This means that in the indicated case a solution of the traveling-wave type exists only when $0 \le s \le s_1$ and, consequently, only for the finite time $0 \le t \le t_1$ (see the dash-dot line in Fig. 1b).

When $\hat{\chi} = -1$ (mass inflow), we obtain $\lambda_0 = -\hat{G}_0$. From Eq. (2.3) it follows that a solution may exist in the entire range of the independent variable $0 \le s \le \infty$ $(0 \le t \le \infty)$ when $\hat{G}_0 < 1 - \nu_0$ and in the finite interval $0 \le s \le s_2$ (i.e., for the finite time $0 \le t \le t_2$) when $\hat{G}_0 > 1 - \nu_0$, where

$$s_2 = 1/\left\{ [\hat{G}_0 - (1 - \nu_0)] (\gamma - 1)^{a_0+1} \xi_1^{a_0} \right\} . \tag{2.4'}$$

Here in the first case $\varphi(\infty) = \infty$ and in the second case $\varphi(s_2) = \infty$ (see Fig. 1c). The instants of time t_1 and t_2 are determined by a formula of the form

$$t_{1,2} = \frac{M_0}{D} s_{1,2} = |\chi_0^{-1}| \, R_0^{a_0} D^{-2a_0} \rho_0^{2a_0 - b_0} s_{1,2} . \tag{2.5}$$

We determine the possible distribution of the function $\xi = \xi(s)$ in the region $s \ge 0$ in each of the above cases using formulas (1.34), (2.2), and (2.3):

a) if $1 - \nu_0 + \lambda_0 = 0$, then in the region $0 \le s \le \infty$ we obtain from (1.34)

$$\xi \equiv \xi_1 = \text{const} , \tag{2.6}$$

i.e., similarly to the case of isentropic flows [19] the entropy is constant both in time and over the space coordinate;

b) if $1 - \nu_0 + \lambda_0 \ne 0$, but $a_0 = 0$, then taking (2.2) into account, we obtain from Eq. (1.34)

$$\xi = \xi_1 \exp(\hat{\chi}_0 (\gamma - 1)(1 - \nu_0 + \lambda_0) s) , \quad \lambda_0 = \hat{G}_0/\hat{\chi}_0 ; \tag{2.7}$$

c) if $a_0(\gamma - 1)(1 - \nu_0 + \lambda_0) = 0$, then substituting (2.3) into (1.34), we obtain

$$\xi = \left[1 - \hat{\chi}_0 (\gamma - 1)^{a_0+1} \xi_1^{a_0} (1 - \nu_0 + \lambda_0) s \right]^{-1/a_0} . \tag{2.8}$$

From Eq. (2.8) it follows that for $\hat{\chi}_0 = 1$ $(\lambda_0 = \hat{G}_0)$ a solution can exist in the entire region $0 \le s \le \infty$ if the condition $\hat{G}_0 < \nu_0 - 1$ is satisfied, with $\xi(\infty) = 0$, and only on the finite interval $0 \le s \le s_1$ for $\hat{G}_0 > \nu_0 - 1$, where the parameter s_1 is defined by formula (2.4). In this case $\xi(s_1) = \infty$. When $\hat{\chi}_0 = -1$ $(\lambda_0 = -\hat{G}_0)$, a solution can exist for $0 \le s \le \infty$ if $\hat{G}_0 < 1 - \nu_0$, with $\xi(\infty) = 0$, and exists only on the finite interval $0 \le s \le s_2$ for $\hat{G}_0 > 1 - \nu_0$.

3. Character of the Distribution of the Dimensionless Function of the Specific Volume and Other Gasdynamic Quantities. We investigate the character of the distribution of the function $\eta = \eta(s)$ for the case where there are volumetric mass losses in a perturbed medium, i.e., for $\hat{\chi}_0 = 1$. We consider a number of typical cases.

1°. Suppose $\hat{G}_0 = \nu_0 - 1 - 2/(\gamma - 1)$. For $\hat{\chi}_0 = 1$ Eq. (1.37) takes the form

$$\varphi \left[\eta^{\gamma+1} - (\gamma - 1)\, \xi_1 \gamma \right] \eta' = -\eta \left[\eta^{\gamma+1} + 2(\gamma - 1)\, \xi_1 \right] \varphi' . \tag{3.1}$$

Introducing the change of variables

$$y = \frac{1}{\gamma(\gamma - 1)\, \xi_1} \eta^{\gamma+1} \tag{3.2}$$

and selecting φ as the independent variable, from Eq. (3.1) we obtain

$$\frac{dy}{d\varphi} = -(\gamma + 1) \frac{y}{\varphi} \frac{y + 2/\gamma}{y - 1} . \tag{3.3}$$

The solution of Eq. (3.3) should satisfy the boundary conditions that correspond to conditions (1.38) at the point $s = 0$ (the traveling-wave front):

$$\varphi = 1 , \quad y = y_1 = \frac{\eta_1^{\gamma+1}}{\gamma(\gamma - 1)\, \xi_1} = \frac{\gamma - 1}{2\gamma} < 1 . \tag{3.4}$$

Equation (3.3) has an analytical solution that allows one to express φ in terms of y. Taking into account boundary condition (3.4), we find

$$\varphi = \left[\left(\frac{y_1 + 2/\gamma}{y + 2/\gamma} \right)^{(\gamma+2)/2} \left(\frac{y}{y_1} \right)^{\gamma/2} \right]^{1/(\gamma+1)} . \tag{3.5}$$

In this case, by virtue of (3.2)

$$\eta = [\gamma(\gamma - 1)\, \xi_1 y]^{1/(\gamma+1)} . \tag{3.6}$$

From formulas (2.2) and (2.3) it follows that the function φ changes within the range $0 \le s \le \infty$, with $\varphi(\infty) = 0$. In the case of $a_0 = 0$, from Eq. (2.2) we obtain

$$s = -\ln \varphi , \quad 0 \le s \le \infty , \tag{3.7}$$

and for $a_0 \ne 0$ from (2.3) we obtain

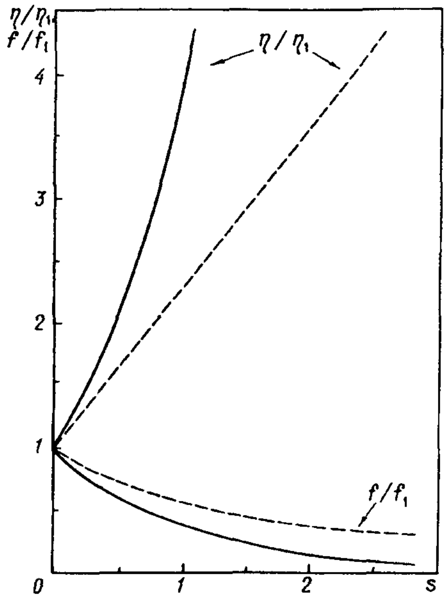
$$s = \frac{1 - \varphi^{2a_0}}{2(\gamma - 1)^{a_0} \xi_1^{a_0} a_0 \varphi^{2a_0}} , \quad 0 \le s \le \infty . \tag{3.8}$$

The functions ξ, f, α, and β can be determined from formulas (1.34), (1.35), and (1.31):

$$\xi = \xi_1 \varphi^2 , \quad f = (\gamma - 1)\, \xi \eta^{-(\gamma-1)} , \quad \alpha = 1 - \varphi\eta , \quad \beta = f\eta^{-1} . \tag{3.9}$$

2°. Suppose $\hat{G}_0 > \nu_0 - 1 - 2/(\gamma - 1)$. An analysis of Eq. (1.37) carried out similarly to the previous one gives the following results.

When $\hat{G}_0 = \nu_0 - 1$, the function $\eta = \eta(s)$ can be represented in implicit form:



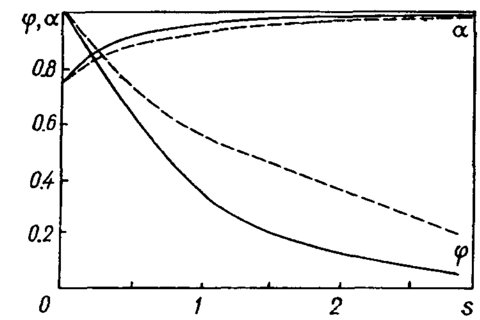


Fig. 2. Distribution of dimensionless specific-volume $\eta = \eta(s)$ and temperature $f = f(s)$ functions referred to the initial values $\eta(0) = \eta_1$, $f(0) = f_1$ at $\gamma = 5/3$, $\hat{\chi}_0 = 1$, $\hat{G}_0 = -3$ for the cases of $a_0 = 0$ (solid lines) and $a_0 = 0.5$ (dashed lines).

Fig. 3. Distribution of the velocity function $\alpha = \alpha(s)$ and the function $\varphi = \varphi(s)$. The notation is the same as in Fig. 2.

$$s = \ln\left[\left(\frac{\eta}{\eta_1}\right)^{\gamma+1} \frac{(\gamma - 1)^2}{(\gamma + 1)^2 \left(\frac{\eta}{\eta_1}\right)^{\gamma+1} - 4\gamma}\right]^{1/[2\chi_0(\gamma-1)^{a_0}\xi_1^{a_0}]}, \tag{3.10}$$

where $\eta_1 = (\gamma - 1)/(\gamma + 1)$. From Eq. (3.10) it follows that in the case considered a solution of the traveling-wave type exists in the entire range of the variable $s \geq 0$ and, consequently, $t \geq 0$. In this case the dimensionless specific volume for $0 \leq s \leq \infty$ changes within the range $\eta_1 \geq \eta \geq \eta_1 [4\gamma/(\gamma + 1)^2]^{1/(\gamma+1)}$.

When $\hat{G}_0 = \nu_0 - 1 + \gamma/(\gamma - 1)$, we obtain

$$\eta = \eta_1 \varphi^{-1}, \quad f = f_1 \varphi^{-1}. \tag{3.11}$$

When $a_0 = 0$, from Eq. (2.2) we have $\varphi = \exp(-s)$. A solution exists in the entire range $s \geq 0$ $(t > 0)$, with $\varphi(\infty) = 0$, $\xi(\infty) = \infty$, $f(\infty) = \infty$, $\eta(\infty) = \infty$, while the pressure β and the velocity α are constant in the entire range of the variable s: $\beta \equiv \beta_1$, $\alpha \equiv a_1$.

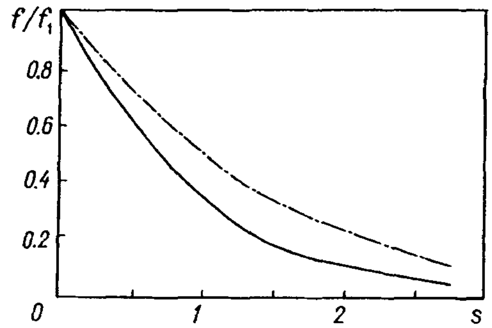
We note that in contrast to the case considered, the solution that describes a traveling wave is fully "constant" at $\chi \equiv 0$, $G \equiv 0$, i.e., when $s \geq 0$, all the sought functions have the form [19]

$$\eta \equiv \eta_1, \quad \alpha \equiv \alpha_1, \quad \beta \equiv \beta_1, \quad f \equiv f_1, \quad \xi \equiv \xi_1. \tag{3.12}$$

When $a_0 \neq 0$, the function $\varphi = \varphi(s)$ is described by formula (2.3). A solution exists only in the finite interval $0 \leq s \leq s_1$, where

$$s_1 = 1/[(\gamma - 1)^{a_0} \xi_1^{a_0} \gamma].$$

When $\hat{G} \neq \nu_0 - 1$ and $\hat{G}_0 \neq \nu_0 - 1 + \gamma/(\gamma - 1)$, an analysis shows that depending on the values of the parameters $\hat{G}_0$ and ν_0 the behavior of the function $\eta = \eta(s)$ and the other gasdynamic quantities can be different



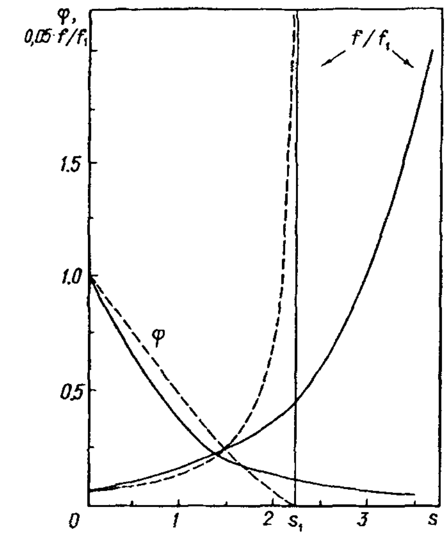


Fig. 4. Distribution of the function $f(s)/f_1$ at $\gamma = 5/3$, $\hat{\chi}_0 = 1$, $\hat{G}_0 = -3$, $a_0 = 0$ for the cases of $\nu_0 = 1$ (solid line) and $\nu_0 = 0$ (dash-dotted line).

Fig. 5. Profiles of the functions $\varphi = \varphi(s)$ and $0.05f(s)/f_1$ at $\gamma = 5/3$, $\hat{\chi}_0 = 1$, $\hat{G}_0 = 2.5$, $\nu_0 = 1$ for the cases of $a_0 = 0$ (solid line) and $a_0 = 0.5$ (dashed line).

in the range $s \geq 0$. In this case a solution may exist both in the range of the independent variable $s \geq 0$ and only in the finite interval $0 \leq s \leq s_1$.

Results of numerical examples that illustrate the characteristic properties of the solutions are presented in Figs. 2-5. All of the examples were considered on the assumption that there are volumetric losses of mass ($\hat{\chi}_0 = 1$) at $\gamma = 5/3$, $b_0 = -a_0(\gamma - 1)$.

The distribution of the quantities presented in Figs. 2-4 was plotted for $\hat{G}_0 = -3$, i.e., for the case of an energy sink. The examples show the influence of the parameter a_0 on the solution, i.e., the exponent of the dependence of the mass sink strength on the temperature. It is seen that at a fixed value of s the temperature is higher and the density is lower for $a_0 = 0.5$ than for $a_0 = 0$. The numerical examples also show that a different technique for allowing for mass losses substantially influences the character of the solution (see Fig. 4).

In the examples given in Figs. 2-4 a solution exists in the entire range of the independent variable $s \geq 0$. Profiles of the dimensionless temperature function $f(s)/f_1$ and the function $\varphi = \varphi(s)$ for $\hat{G}_0 = 2.5$ (energy inflow) and $\nu_0 = 1$ are presented in Fig. 5. In the indicated case at $a_0 = 0$ a solution exists in the range $0 \leq s \leq \infty$, with $f(\infty) = \infty$, $\varphi(\infty) = 0$. When $a_0 = 0.5$, a solution exists only in the finite interval $0 \leq s \leq s_1$. In the specific example considered, $s_1 = 2.2$. The region of existence of the solution in time is confined within the interval $0 \leq t \leq t_1$, where $t_1 = 2.2M_0/D = 2.2\chi_0^{-1}R_0^{1/2}D^{-1}\rho_0^{4/3}$.

It was noted above that under certain conditions a solution of the traveling-wave type exists only over a finite interval of the independent variable s and, consequently, only for a finite time $0 \leq t \leq t_1$. In particular, under certain conditions the sought solution exists until the following condition (see Eq. (1.37)) is satisfied:

$$\frac{1}{\gamma(\gamma-1)\xi_1}\eta^{\gamma+1}\varphi^{(\gamma-1)(1-\nu_0+\lambda_0)+2} = 1\,. \tag{3.13}$$

Using formula (1.34), relation (3.13) is written as

$$\varphi^2 = \gamma(\gamma-1)\xi\eta^{-\gamma-1} \tag{3.14}$$

or by virtue of (1.35)

$$\varphi^2 = \gamma\beta\eta^{-1}\,. \tag{3.15}$$

We go over to the initial dimensional variables by formulas (1.20). Relation (3.15) takes the form

$$\psi^2 = \frac{\gamma P \rho}{D^2}. \tag{3.16}$$

It is known from gas dynamics [19] that in the case where the equations of state of an ideal gas (1.10) are valid, the combination $\gamma P\rho$ determines the square of the "mass" speed of sound $C_M = \rho C_\gamma$, where $C_\gamma = \sqrt{\gamma P/\rho}$. Taking the latter into account, we obtain the following expression from (3.16):

$$\psi^2 D^2 = C_M^2. \tag{3.17}$$

By definition, $\psi = \partial m/\partial t$. The mass velocity of the traveling-wave front at the point considered can be determined from the relation

$$D_M = \frac{\partial m\,(q,\,t)}{\partial t} = \frac{\partial m}{\partial q}\frac{\partial q}{\partial t} = \psi D. \tag{3.18}$$

With allowance for (3.18), formula (3.17) is written as

$$D_M^2 = C_M^2. \tag{3.19}$$

Thus, at the point where condition (3.13) is satisfied, the mass velocity of the traveling wave coincides with the local speed of sound. Here, in contrast to the case where mass sinks or sources are not taken into account (see [7, 11-14]), the mass velocity of the traveling-wave front is not constant: it is proportional to the function $\psi = \psi(q, t)$, i.e., to the fraction of the mass left or acquired in the considered element of the flow.

NOTATION

t, time; r, Euler spatial coordinate; t_L, quasi-Lagrangian time coordinate; m, Lagrangian mass coordinate; q, quasi-Lagrangian mass coordinate; ρ, density; v, velocity; P, pressure; T, temperature; ε, specific internal energy; $\chi = \chi(\rho, T)$ and $G = G(\rho, T)$, strengths of volumetric mass and energy sinks ($\chi > 0$, $G < 0$) or sources ($\chi < 0$, $G > 0$), respectively; $\psi = \psi(q, t)$, fraction of the mass left or acquired in the given element of the flow; s, dimensionless "self-similar" variable; $\eta = \eta(s)$, $\alpha = \alpha(s)$, $\beta = \beta(s)$, $f = f(s)$, $\hat{\varepsilon} = \hat{\varepsilon}(s)$, dimensionless specific volume, velocity, pressure, temperature, and specific internal energy functions of the variable s, respectively; $\Sigma = \Sigma(q, t)$ and $\xi = \xi(s)$, dimensional and dimensionless "entropy" functions, respectively; a_0 and b_0, exponents of the dependence of the strength of the mass source and sink on the temperature and density; $\hat{\chi}_0 = 1$ for the case of a mass sink; $\hat{\chi}_0 = -1$ for the case of mass inflow at the given element of the flow; $\hat{G}_0$, dimensionless constant in the formula that expresses the strength of the energy source ($\hat{G}_0 > 0$) or sink ($\hat{G}_0 < 0$).

REFERENCES

1. L. N. Busurina, P. P. Volosevich, G. G. Zukakishvili, G. S. Latsabidze, E. I. Levanov, Yu. P. Popov, N. M. Skhirtladze, and E. K. Tikhanov, Fiz. Plazmy, 8, No. 5, 1053-1062 (1982).
2. P. P. Volosevich, I. I. Galiguzova, N. A. Dar'in, G. G. Zukakishvili, V. Ya. Karpov, E. I. Levanov, R. G. Salukvadze, and E. K. Tikhanov, Computational Experiments on Theta-Pinches with Account for End-Face Losses, Preprint No. 127, Inst. Prikl. Mat. im. M. V. Keldysha Akad. Nauk SSSR, Moscow (1984).
3. P. P. Volosevich, M. A. Kornilina, E. I. Levanov, and G. V. Meladze, Effect of End-Face Losses on Compression and Heating in Theta-Pinches, Preprint No. 195, Inst. Prikl. Mat. im. M. V. Keldysha Akad. Nauk SSSR, Moscow (1958).

4. P. P. Volosevich, N. A. Dar'in, V. Ya. Karpov, and A. Yu. Krukovskii, Calculation of Problems of Magnetic Hydrodynamics with Mass Sinks in Quasi-Lagrangian Coordinates, Preprint No. 7, Inst. Prikl. Mat. im. M. V. Keldysha Akad. Nauk SSSR, Moscow (1984).
5. P. P. Volosevich and E. I. Levanov, in: Mathematical Simulation. Current Problems of Mathematical Physics and Computational Mathematics [in Russian], Moscow (1989), pp. 42-88.
6. P. P. Volosevich, M. A. Kornilina, E. I. Levanov, and G. V. Édzhibia, Matem. Modelir., 5, No. 2, 25-41 (1993)
7. P. P. Volosevich, N. A. Dar'in, E. I. Levanov, and N. M. Skhirtladze, Problem of a Piston in a Gas with Sources and Sinks (Self-Similar Solution) [in Russian], Tbilisi (1986).
8. L. N. Busurina, P. P. Volosevich, E. I. Levanov, G. V. Meladze, and G. V. Édzhiba, Self-Similar Problems of Magnetic Hydrodynamics with Account for Sources (Sinks) of Various Kinds, Preprint No. 10, Inst. Prikl. Mat. im. M. V. Keldysha Akad. Nauk SSSR, Moscow (1992).
9. L. A. Artsimovich, Controlled Thermonuclear Reactions [in Russian], Moscow (1963).
10. E. G. Gamalii, S. Yu. Gus'kov, O. N. Krokhin, and V. B. Rozanov, Fiz. Plazmy, 1, No. 6, 904-911 (1975).
11. A. A. Samarskii, S. P. Kurdyumov, and P. P. Volosevich, Zh. Vychisl. Mat. Mat. Fiz., 5, No. 2, 199-217 (1965).
12. S. P. Kurdyumov, Study of the Interaction of Hydrodynamic and Nonlinear Thermal Processes Using Traveling Waves, Preprint No. 55, Inst. Prikl. Mat. im. M. V. Keldysha Akad. Nauk SSSR, Moscow (1971).
13. S. P. Kurdyumov, Traveling Temperature-Hydrodynamic Waves Moving over the Background with a Constant Pressure, Preprint No. 45, Inst. Prikl. Mat. im. M. V. Keldysha Akad. Nauk SSSR, Moscow (1971).
14. P. P. Volosevich, E. I. Levanov, and V. I. Maslyankin, Self-Similar Problems of Gas Dynamics [in Russian], Moscow (1984).
15. A. A. Samarskii and Yu. P. Popov, Difference Methods of Solving Problems of Gas Dynamics [in Russian], Moscow (1980).
16. Yu. A. Poveshchenko and Yu. P. Popov, Zh. Vychisl. Mat. Mat. Fiz., 18, No. 4, 1048-1056 (1978).
17. A. V. Koldoba and Yu. A. Poveshchenko, Fully Conservative Difference Schemes for Problems of Gas Dynamics in the Presence of Mass Sources, Preprint No. 160, Inst. Prikl. Mat. im. M. V. Keldysha Akad. Nauk SSSR, Moscow (1982).
18. L. I. Sedov, Similarity and Difference Methods in Mechanics [in Russian], Moscow (1981).
19. L. D. Landau and E. M. Lifshits, Mechanics of Continua [in Russian], Moscow (1954).